Lecture 27

In this lecture we continue our study of counting methods, looking at the Binomial Theorem and the Pigeonhole Principal.

The Binomial Theorem

Consider the set of all binary strings (or "words") with 3 bits.

In Coding Theory and Language Theory it's often important to identify the different <u>weights</u> among binary words with some fixed number of bits. The weight $w(\underset{\sim}{x})$ of a binary word $\underset{\sim}{x}$ is the number of times that 1 appears in the word.

E.g., $w(101) = 2$

Among 3-bit words, what weights occur and how many different words are there having any specified weight?

Of course, the same question can be asked about 1-bit, 2-bit, 4-bit words (etc).

Among n-bit words, how many have weight r (where $0 \le r \le n$) ? The answer is ${}^nC_r$ — the number of ways to select the r positions where 1 is to occur.

1 bit	weight	count
0	0	1
1	1	1

2 bits	weight	count
00	0	1
10	1	2
01	1	
11	2	1

3 bits	weight	count
000	0	1
100	1	3
010	1	
001	1	
110	2	3
101	2	
011	2	
111	3	1

When $n=1$ we get:

$$1, 1$$

When $n=2$ we get:

$$1, 2, 1$$

When $n=3$ we get:

$$1, 3, 3, 1$$

Leaving out the commas, we get several familiar rows of numbers. If we insert an extra row at the top, consisting of a single 1, we get the famous pattern known as Pascal's Triangle:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \\ &&&& \vdots &&&& \end{array}$$

Every number below the first row is the sum of the two numbers closest to it in the row above, one number being above and just to the left while the other number is above and just to the right.

The next row would be:

1 5 10 10 5 1

Because of the relationship between Pascal's triangle and the binomial theorem, from this new row we can deduce that

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + \\ + 10x^2y^3 + 5xy^4 + y^5 .$$

The binomial theorem has many applications.

For example, it can be used to produce approximations.

<u>E.g.</u>

Estimate $(2.01)^3$ to 2 decimal places.

<u>Solution</u>

$$
\begin{aligned}
(2.01)^3 &= (2 + .01)^3 \\
&= 2^3 + 3\cdot 2^2\cdot(.01) + 3\cdot 2\cdot(.01)^2 + \\
&\qquad + (.01)^3 \\
&= 8 + .12 + \underbrace{6(.0001) + (.000001)}_{\text{terms whose sum is } < .001} \\
&\approx 8.12
\end{aligned}
$$

The rows of Pascal's triangle appear in a wide variety of situations.

Expanding the RHS gives:

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \cdots$$

$$+ {}^nC_{n-2}\, x^2 y^{n-2} + {}^nC_{n-1}\, x y^{n-1} + {}^nC_n\, y^n$$

$$= x^n + n x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \cdots$$

$$+ {}^nC_2\, x^2 y^{n-2} + n x y^{n-1} + y^n$$

For $n=1$ this becomes

$$(x+y)^1 = x^1 + y^1.$$

Less trivially, for $n=2$ we get the familiar formula

$$(x+y)^2 = x^2 + 2xy + y^2.$$

For $n=3$ the result is

$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3 ;$$

and so on.

Let's look at the lists of binomial coefficients.

In general, the set of all n-bit words will have ${}^nC_0 (= 1)$ word of weight 0, ${}^nC_1$ words of weight 1, ${}^nC_2$ words of weight 2, and so on.

It will have ${}^nC_{n-1} (= {}^nC_1 = n)$ words of weight $(n-1)$, and ${}^nC_n (= {}^nC_0 = 1)$ word of weight n.

These numbers

$${}^nC_0,\ {}^nC_1,\ \ldots,\ {}^nC_n$$

are called binomial coefficients. They appear in the following well known theorem.

<u>The Binomial Theorem</u>

For every non-negative integer n,

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k\, x^{n-k} y^k .$$

Here's a useful corollary to the binomial theorem.

Theorem

For any non-negative integer n,

$$\sum_{k=0}^{n} {}^{n}C_{k} = 2^{n}.$$

Proof

Putting $x = y = 1$ in the binomial theorem gives

$$(1+1)^{n} = \sum_{k=0}^{n} {}^{n}C_{k}\, 1^{n-k}\, 1^{k}$$

i.e., $$2^{n} = \sum_{k=0}^{n} {}^{n}C_{k}.$$

Applying this to the case where $n = 3$ gives:

$${}^{3}C_{0} + {}^{3}C_{1} + {}^{3}C_{2} + {}^{3}C_{3} = 2^{3}$$

i.e., $$1 + 3 + 3 + 1 = 8$$

Now if we look back at the question of the weights of the 3-bit binary words, we can see how the numbers arise.

There are $2^3 = 8$ words that have 3 bits. Of those, ${}^3C_0 (= 1)$ has weight 0, ${}^3C_1 (= 3)$ have weight 1, ${}^3C_2 (= {}^3C_1 = 3)$ have weight 2 and ${}^3C_3 (= {}^3C_0 = 1)$ has weight 3.

Now we move on, and look at an apparently obvious and yet exceedingly useful principle which is often applied in situations where we want to count things.

The Pigeonhole Principle

If $(n+1)$ or more pigeons are put into n holes, at least two pigeons must be in the same hole.

E.g. Suppose that we have to put numbers in squares. We know that in the first square we can have 1 or 2, in the second square we can have 1 or 2, and in the third square we can have 1, 2 or 3. But each square must have a different number in it. Can we put any of the numbers in any of the squares with certainty?

This problem often arises in a game of Sudoku.

1 2	1 2	1 2 3	Options

The answer is that we must have 3 in the third box.

If we put 1 in there, then we cannot have 1 in either of the first two boxes. So we have one number (which is 2) available for two boxes. The Pigeonhole Principle says that both boxes must end up with the same number, which violates the rules of the game. Putting 2 in the third box has the same effect. So only 3 can go in the third box.

In this example the "pigeons" are the first two boxes, and there is only one "pigeonhole" — the number 2.